

ALTA

Relative Positional Accuracy

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Agenda

1. Relative Positional Accuracy
2. The Mathematical Test
3. Statistical Review
4. Computing The Error Ellipse
5. Sample Network Projects
6. Network Design
7. Tips For Success

1. Relative Positional Accuracy

“Relative Positional Accuracy (RPA) is the value expressed in feet or meters that represents the uncertainty due to random errors in measurements in the location of any point on a survey to any other point on the same survey, at the 95 percent confidence level.” *Rule 12, 865 IAC, 1-12-2.*

The ALTA (American Land Title Association) test is based on the relative error ellipse and the horizontal distance between each station pair. The relative error ellipse represents the relative precision of each station pair. This test was jointly developed by ALTA and the American Congress on Surveying and Mapping (ACSM).

RPA may be tested by:

1. Comparing the relative location of points in a survey as measured by an independent survey of higher accuracy.
2. Testing the results from a minimally-constrained, correctly weighted, least squares adjustment of the survey.

1. Relative Positional Accuracy

WHY Least Squares

1. Distributes the error following laws of random probability.
2. Only way to handle complex networks.
3. Minimizes the sum of squared residuals.
4. The best any other method can do is to equal the results of a least squares adjustment.

REQUIREMENTS for valid results

1. Instruments properly tuned.
2. Proper field procedures used.
3. Blunders and systematic errors removed or minimized.
4. Remaining inaccuracy is due to random errors.

PROCEDURE

1. Minimally-constrained adjustment tied to one station + a direction (azimuth, bearing or a vector).
2. Proper weighting employed by assigning expected errors (standard deviations) to observations.

2. The Mathematical Test

1. Determine the relative (not absolute) error ellipse between all station pairs in the survey. Each ellipse has a semi-major and semi-minor axis; use the semi-major axis (the larger of the two).
2. A four-station network results in six relative error ellipses.

A 10-station network has 45 relative error ellipses.

A 50-station network has 1225 relative error ellipses.

Number of relative error ellipses is:

$$n \times (n - 1) / 2$$

where n = number of stations.

2. The Mathematical Test

Matrix showing combinations for station 1, 2, 3, 4

	1	2	3	4
1	A			
2	R	A		
3	R	R	A	
4	R	R	R	A

A: Absolute error ellipse (not used for ALTA).

R: Relative error ellipse between each station pair.

2. The Mathematical Test

3. Scale the semi-major ellipse to the 95% confidence level using the K value of 2.448.
4. Compute the total allowable uncertainty for each station pair using the fixed tolerance and the variable PPM (Parts Per Million).

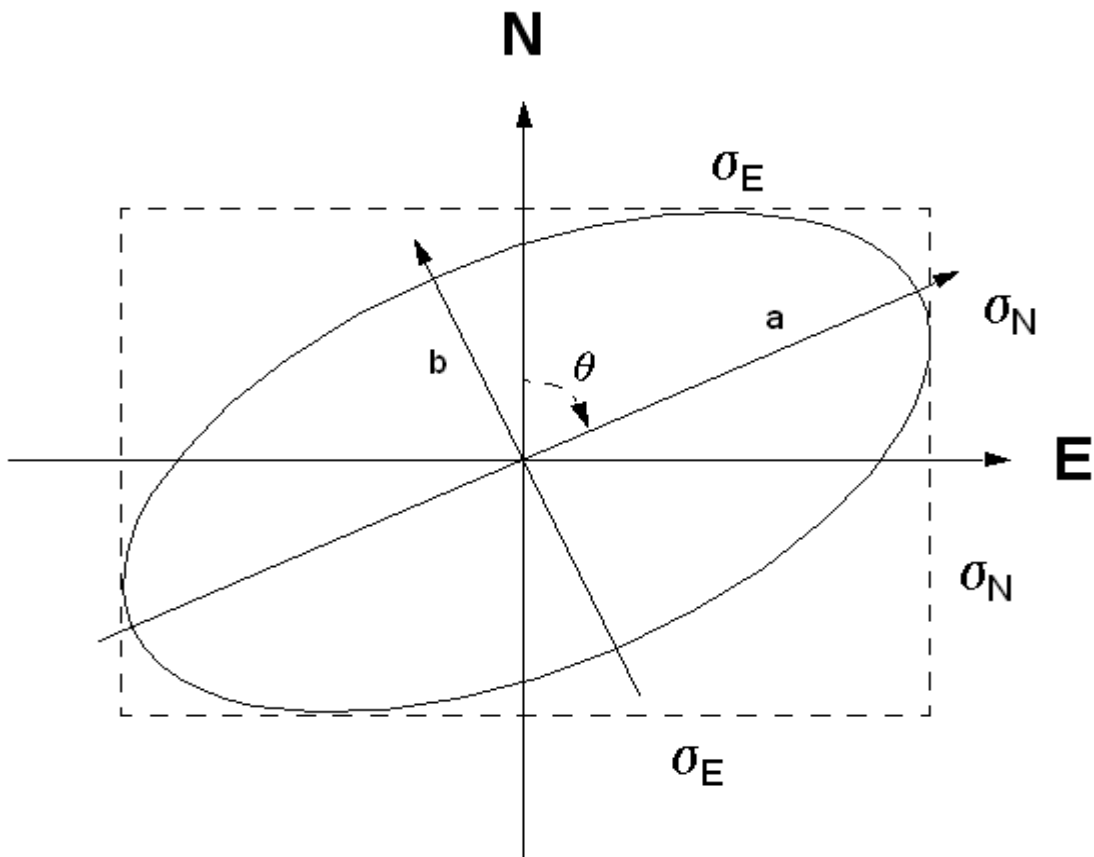
Total Tolerance Allowed
0.07 feet + 50 PPM

The PPM contribution increases with increasing horizontal distance between any two station pairs.

5. The ratio of the scaled relative error ellipse to the Total Tolerance Allowed must be ≤ 1.0 .

2. The Mathematical Test

This is a graphical representation of the error ellipse. The semi-major ('a') and semi-minor ('b') axes are computed. The ellipse is then rotated clockwise from the north to the direction given by the azimuth (θ).



2. The Mathematical Test

EXAMPLE 1: Between Stations 1 and 2

Relative standard error ellipse

0.035 feet (semi-major axis)

Scaled to 95% confidence level ($K = 2.448$)

$0.035 \times 2.448 = 0.086$ feet

Horizontal distance between 1 and 2

250.0 feet

Fixed allowable error

0.07 feet

Variable allowable error (50 ppm)

$50 \times 250 \text{ feet} / 1000000 = 0.0125$

Total allowable error

$0.07 + 0.0125 = 0.0825$

Scaled error ellipse / total allowable error

$0.086 / 0.0825 = 1.042$ ($1.042 > 1.0$, Test Fails)

2. The Mathematical Test

EXAMPLE 2: Between Stations 3 and 4

Relative standard error ellipse

0.037 feet (semi-major axis)

Scaled to 95% confidence level ($K = 2.448$)

$0.037 \times 2.448 = 0.091$ feet

Horizontal distance between 3 and 4

1000.0 feet

Fixed allowable error

0.07 feet

Variable allowable error (50 ppm)

$50 \times 1000 \text{ feet} / 1000000 = 0.050$

Total allowable error

$0.07 + 0.050 = 0.120$

Scaled error ellipse / total allowable error

$0.091 / 0.120 = 0.758$ ($0.758 < 1.0$, Test Passes)

3. Statistical Review

STATISTICS

A way to convert numbers into useful information so that good decisions can be made.

We use statistics to develop confidence in a survey. We can also use statistics to find problems with a survey.

DESCRIPTIVE Statistics

- Baseball batting averages.
- Basketball average points scored per game.
- Measuring a distance 10 times to determine the mean and dispersion.

INFERENCEAL Statistics

- Make claims or conclusions about population based on a sample of data from that population.
- Population represents all possible outcomes.
- Sample is a subset of a population.
- Inferential statistic requires mathematical models that involve probability theory.
- You have to make these decisions.

3. Statistical Review

DESCRIPTIVE Statistics

A height difference between Stations 1 and 2, measured 15 times. Weight is the same for all measurements.

Num	Value	Weight	Value * Weight
1	9.98	1.0	9.98
2	9.99	1.0	9.99
3	10.01	1.0	10.01
4	10.01	1.0	10.01
5	10.01	1.0	10.01
6	10.01	1.0	10.01
7	10.02	1.0	10.02
8	10.02	1.0	10.02
9	10.02	1.0	10.02
10	10.02	1.0	10.02
11	10.02	1.0	10.02
12	10.03	1.0	10.03
13	10.03	1.0	10.03
14	10.04	1.0	10.04
15	10.05	1.0	10.05

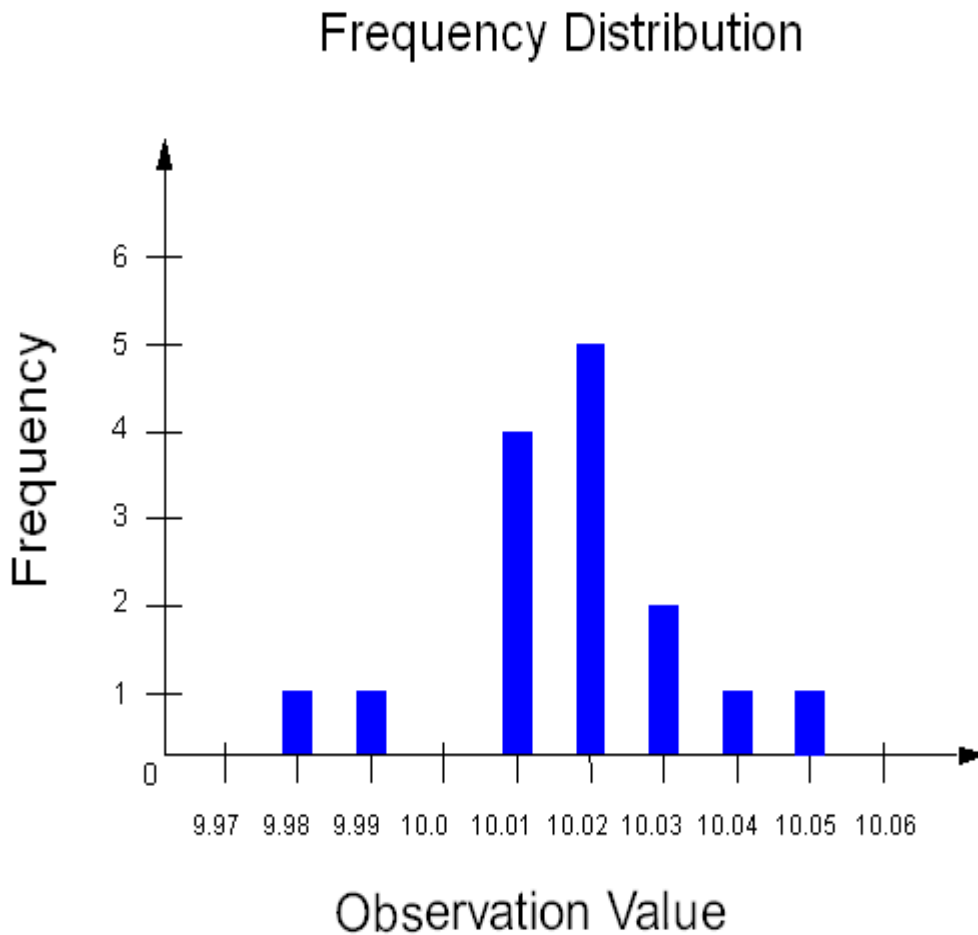
3. Statistical Review

Same data, but tabulated by frequency of occurrence.
 Weight is the same for all measurements.

Frequency	Value	Freq * Value	Weight	Freq * Val * Wgt
1	9.98	9.98	1.0	9.98
1	9.99	9.99	1.0	9.99
4	10.01	40.04	1.0	40.04
5	10.02	50.10	1.0	50.10
2	10.03	20.06	1.0	20.06
1	10.04	10.04	1.0	10.04
1	10.05	10.05	1.0	10.05
15			15.0	150.26
			Mean	150.26 / 15.0
			Mean	10.0175

3. Statistical Review

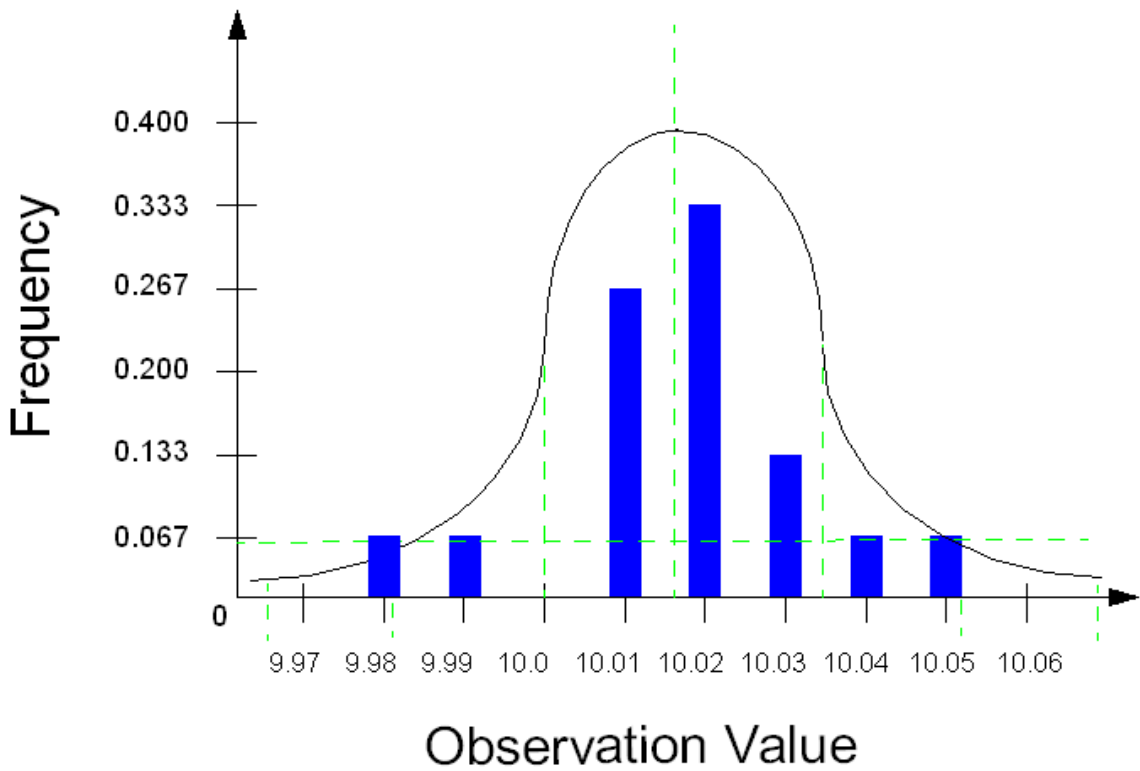
Graphical representation of the height difference data. Notice how it tends to form a bell-shaped curve centered around the mean of 10.0175.



3. Statistical Review

Same data shown in relative frequency form. The area under this curve is 1.0 (100% of probability). Notice that 10.02 occurs 33.3% of the time. The curve follows the pattern of the normal probability distribution.

Relative Frequency Distribution Curve



3. Statistical Review

CALCULATING Descriptive Statistics

Measures of Central Tendency (center point of data set):

Mean - Arithmetic mean

Median - Mid Point Value (half below, half above)

Mode - Most common value

X_i - i 'th observation value

W_i - i 'th observation weight (all 1.0 in this example)

Mean - (summation all $X_i * W_i$) / (summation all W_i)

- 150.26 / 15.0

- 10.017

Median - 10.02

Mode - 10.02

The Mean is the least squares solution. In this example, 10.017 results in the minimization of the sum of squared weighted residuals, where the weight for each observation is 1.0.

3. Statistical Review

Obs	Xi	Wi	Xi * Wi	Mean	Residual	(Residual) ²
1	9.98	1.0	9.98	10.017	0.037	0.001369
2	9.99	1.0	9.99	10.017	0.027	0.000729
3	10.01	1.0	10.01	10.017	0.007	0.000049
4	10.01	1.0	10.01	10.017	0.007	0.000049
5	10.01	1.0	10.01	10.017	0.007	0.000049
6	10.01	1.0	10.01	10.017	0.007	0.000049
7	10.02	1.0	10.02	10.017	-0.003	0.000009
8	10.02	1.0	10.02	10.017	-0.003	0.000009
9	10.02	1.0	10.02	10.017	-0.003	0.000009
10	10.02	1.0	10.02	10.017	-0.003	0.000009
11	10.02	1.0	10.02	10.017	-0.003	0.000009
12	10.03	1.0	10.03	10.017	-0.013	0.000169
13	10.03	1.0	10.03	10.017	-0.013	0.000169
14	10.04	1.0	10.04	10.017	-0.023	0.000529
15	10.05	1.0	10.05	10.017	-0.033	0.001089
Total	150.26	15.0	150.26		~0.0	~0.004295

3. Statistical Review

Variance and Standard Deviation for the set of observations:

$$s^2 = \text{sample variance} = 0.004295 / (n - 1)$$

$$s^2 = \text{sample variance} = 0.004295 / (14)$$

$$s^2 = \text{sample variance} = 0.000307$$

$$s = \text{sample standard deviation} = \text{sqrt}(s^2)$$

$$s = \text{sample standard deviation} = \text{sqrt}(0.000307)$$

$$s = \text{sample standard deviation} = 0.0175$$

$(n - 1)$ is the redundancy in this set of sample measurements.

3. Statistical Review

THE EMPIRICAL RULE

If a one-dimensional distribution follows a bell-shaped curve – symmetrical curve centered around the mean – we would expect approximately 68, 95 and 99.7 percent of the values to fall within one, two and three standard deviations around the mean, respectively (actual values are 1.0, 1.96 and 2.97, respectively).

Therefore: mean = 10.017, s = 0.0175

$$10.017 - (1.0 \times 0.0175) \leq 68\% \leq 10.017 + (1.0 \times 0.0175)$$

9.999 ≤ 68% ≤ 10.035

$$10.017 - (2.0 \times 0.0175) \leq 95\% \leq 10.017 + (2.0 \times 0.0175)$$

9.982 ≤ 95% ≤ 10.052

$$10.017 - (3.0 \times 0.0175) \leq 99.7\% \leq 10.017 + (3.0 \times 0.0175)$$

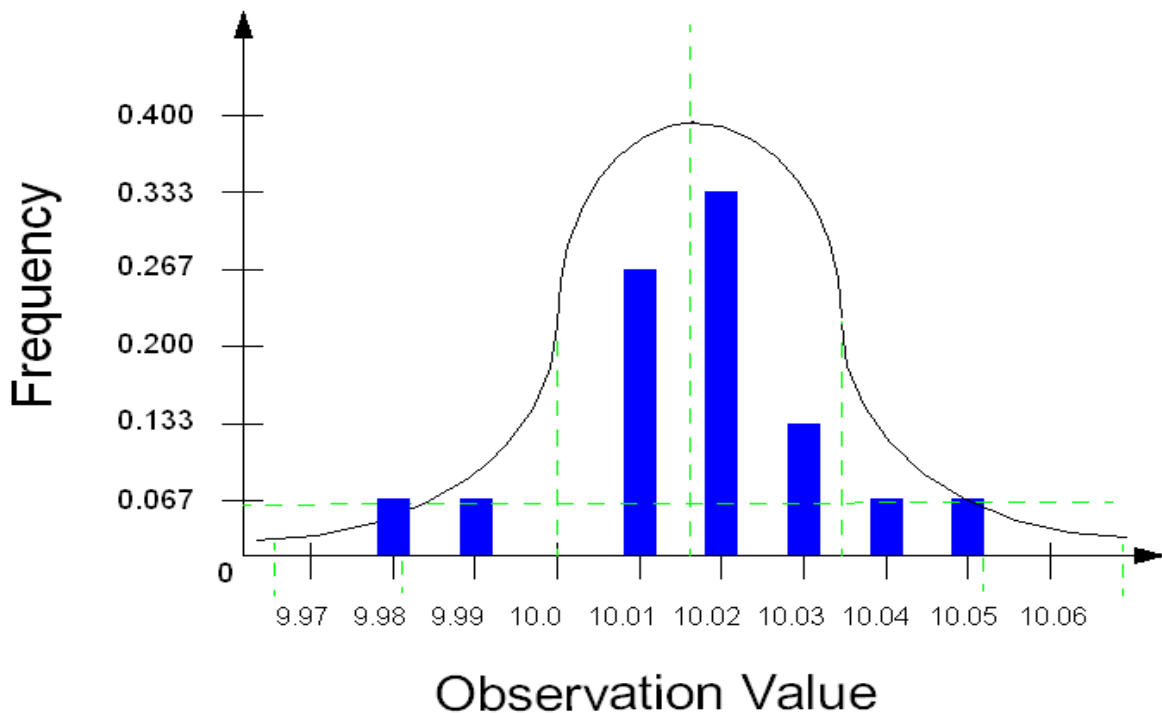
9.965 ≤ 99.7% ≤ 10.070

The error ellipse is two-dimensional. The K value is 2.448 for the 95% confidence level.

3. Statistical Review

Here is the distribution curve that goes with this data. The area under the curve approaches 1.0. The left and right ends never touch the X axis. The curve follows the pattern of the normal probability distribution. The smaller the standard deviation, the narrower the curve.

Relative Frequency Distribution Curve



3. Statistical Review

The mean of all sample observations is the best-fit, least squares solution.

STANDARD deviation of the mean

$$s \text{ mean} = s / \text{sqrt}(\text{num observations})$$

$$s \text{ mean} = 0.0175 / \text{sqrt}(15.0)$$

$$s \text{ mean} = 0.004518$$

VARIANCE of the mean

$$s^2 \text{ mean} = s \text{ mean} * s \text{ mean}$$

$$s^2 \text{ mean} = \sim 0.000020417$$

This is what you would expect if you measured the same height difference 15 times, each day, for 10 or 20 days. The mean of each day's work would follow a normal distribution. The standard deviation and variance of these means would be much smaller than that for an individual set of observations.

If we load this data into an adjustment, we will see this result for the adjusted height standard deviation (LevelNet.txt).

3. Statistical Review

ABSOLUTE Error Ellipse

The absolute standard error ellipse is a representation of a probability. We really do not know in absolute terms where the point we measured really is; we only have a probability of where the point is.

A standard error ellipse (un-scaled) represents a 60.6% chance that the actual point is outside the computed position and the area bounded by the standard error ellipse. To ensure a smaller probability that the point is outside this region, the size of the standard error ellipse must be increased.

By expanding the standard error ellipse by the K value of 2.448, we create a 5% probability that the point may fall outside the computed position and the area bounded by the standard error ellipse. Put another way, the point has a 95% probability of falling within this region, the standard error ellipse is scaled by 2.448.

3. Statistical Review

RELATIVE Error Ellipse

The standard relative error ellipse is a representation of the relative precision of each station pair coordinate difference. This ellipse is created by differencing the coordinate errors for each station pair. Errors common to both stations are eliminated.

The standard relative error ellipse can be smaller than the absolute (station) error ellipse on each end. The coordinates for each station could be completely wrong (e.g., based on incorrectly used fixed coordinates), but the relative errors between stations give the best estimate of the precision of the survey – regardless of the coordinates.

Think in terms of GPS measurements. The station coordinates determined using GPS may be off by meters, but the vector (the difference between these coordinates) can be accurate to the centimeter level or better. The error in this vector is the best indicator as to the quality of the measurement.

To scale the standard relative error ellipse to the 95% confidence level, use a K value of 2.448.

3. Statistical Review

TYPE I Error

In the previous discussion, we implied that we have a 5% probability that the actual point falls outside the computed coordinate value and the bounding error ellipse. There is a 95% chance that the value falls inside this region.

H_0 is the Null Hypothesis (the hypothesis that the true position of the coordinate falls within the computed coordinate value and the bounding error ellipse).

A **Type I** error is committed when H_0 is rejected based on a statistical test of sample data when, in fact, H_0 is true. The probability of committing this type of error is 5% (or alpha), when the confidence level is set to 95%.

Suppose the computer-generated statistics tell us to reject H_0 . We would reject the result, even though it is correct. Fortunately, this only has a 5% chance of occurring. If we reject H_0 when it was really true, we may need additional field work to confirm that the original data, is in fact, acceptable data.

3. Statistical Review

TYPE II Error

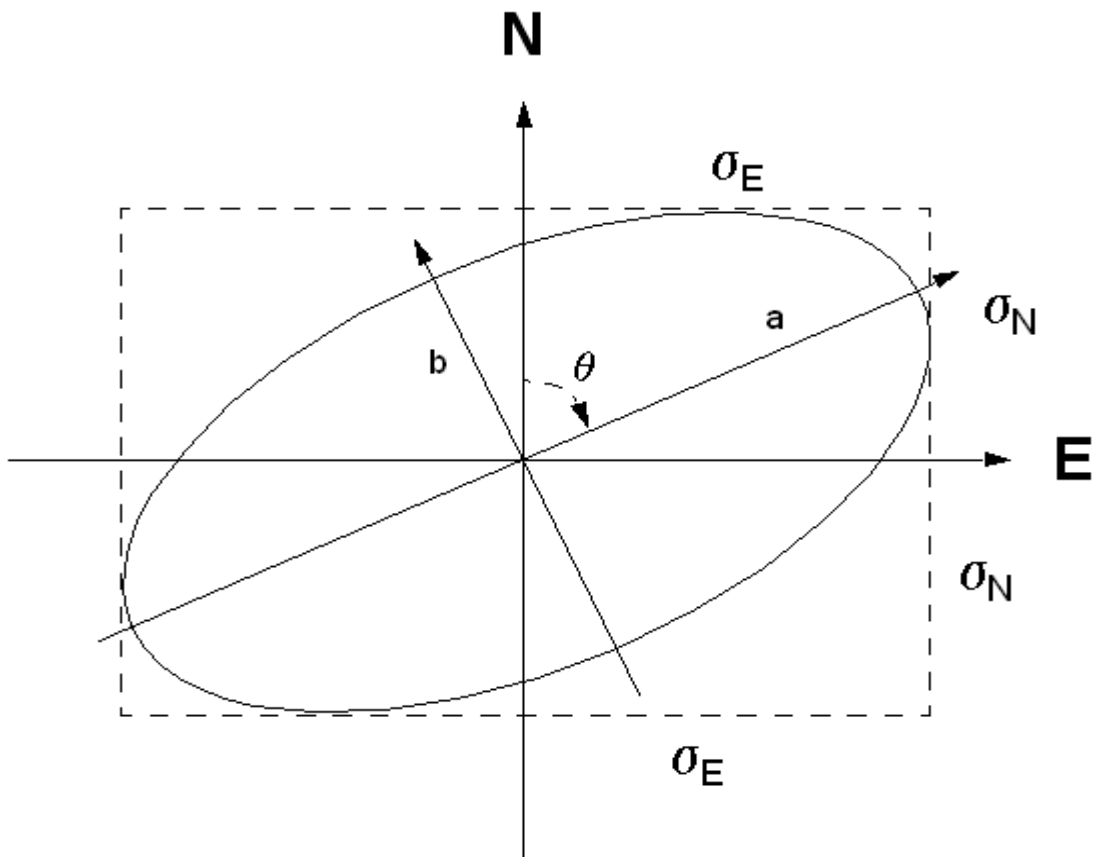
A **Type II** error is committed when H_0 hypothesis is not rejected when, in fact, it is false.

The probability of committing this type of error is difficult to compute, since we will never know what the true coordinate value is.

	H_0 Is True	H_0 Is False
Reject H_0	Type I Error	Correct Outcome
Do Not Reject H_0	Correct Outcome	Type II Error

4. Computing The Error Ellipse

This is a graphical representation of the error ellipse. The semi-major ('a') and semi-minor ('b') axes are computed. The ellipse is then rotated clockwise from the north to the direction given by the azimuth (θ).



4. Computing The Error Ellipse

After a network adjustment, the matrix containing statistical results is generated. For a 2D network, each station has two columns and two rows. For a 3D network, there are three columns and three rows for each station; however, the third row/column (the height row/column) is not used to compute the error ellipse.

	1	2	3	4
1	σ^2_N σ_{NE} σ_{NE} σ^2_E			
2	σ_{N1N2} σ_{E1N2} σ_{N1E2} σ_{E1E2}	σ^2_N σ_{NE} σ_{NE} σ^2_E		
3	σ_{N1N3} σ_{E1N3} σ_{N1E3} σ_{E1E3}	σ_{N2N3} σ_{E2N3} σ_{N2E3} σ_{E2E3}	σ^2_N σ_{NE} σ_{NE} σ^2_E	
4	σ_{N1N4} σ_{E1N4} σ_{N1E4} σ_{E1E4}	σ_{N2N4} σ_{E2N4} σ_{N2E4} σ_{E2E4}	σ_{N3N4} σ_{E3N4} σ_{N3E4} σ_{E3E4}	σ^2_N σ_{NE} σ_{NE} σ^2_E

4. Computing The Error Ellipse

To compute the absolute error ellipse, we need data from the station 2x2 matrix.

Formulas for the absolute error ellipse

$$t_1 = (\sigma_N^2 + \sigma_E^2)$$

$$t_2 = (\sigma_E^2 - \sigma_N^2)$$

$$t_3 = \sigma_{NE}$$

$$t_4 = (\sigma_N^2 - \sigma_E^2)$$

a, b, az require decorrelation of the the 2x2 matrix.

$$a = [1/2 t_1 + (1/4 t_2^2 + t_3^2)^{1/2}]^{1/2}$$

$$b = [1/2 t_1 - (1/4 t_2^2 + t_3^2)^{1/2}]^{1/2}$$

$$\text{angle} = \text{atan2}(2 t_3, t_4)$$

$$\text{az} = \text{convert_to_az}(\text{angle}) / 2$$

a = semi-major axis of ellipse

b = semi-minor axis of ellipse

az = azimuth of orientation

Note: All inverse entries are multiplied by the “**a posteriori variance factor**” before use in any calculation.

4. Computing The Error Ellipse

To compute the relative error ellipse, we need data from each station 2x2 matrix, plus data that describes additional coordinate error differences of the station pair.

Formulas for the relative error ellipse

$$\sigma_{\Delta N}^2 = \sigma_{N1}^2 - 2\sigma_{N1N2} + \sigma_{N2}^2$$

$$\sigma_{\Delta E}^2 = \sigma_{E1}^2 - 2\sigma_{E1E2} + \sigma_{E2}^2$$

$$\sigma_{\Delta N\Delta E} = \sigma_{N1E1} - \sigma_{N1E2} - \sigma_{E1N2} + \sigma_{N2E2}$$

Identical to absolute error ellipse equations

$$t_1 = (\sigma_{\Delta N}^2 + \sigma_{\Delta E}^2)$$

$$t_2 = (\sigma_{\Delta E}^2 - \sigma_{\Delta N}^2)$$

$$t_3 = \sigma_{\Delta N\Delta E}$$

$$t_4 = (\sigma_{\Delta N}^2 - \sigma_{\Delta E}^2)$$

Equations for a, b, az (same as for absolute error ellipse)

$$a = [1/2 t_1 + (1/4 t_2^2 + t_3^2)^{1/2}]^{1/2}$$

$$b = [1/2 t_1 - (1/4 t_2^2 + t_3^2)^{1/2}]^{1/2}$$

$$\text{angle} = \text{atan2}(2 t_3, t_4)$$

$$\text{az} = \text{convert_to_az}(\text{angle}) / 2$$

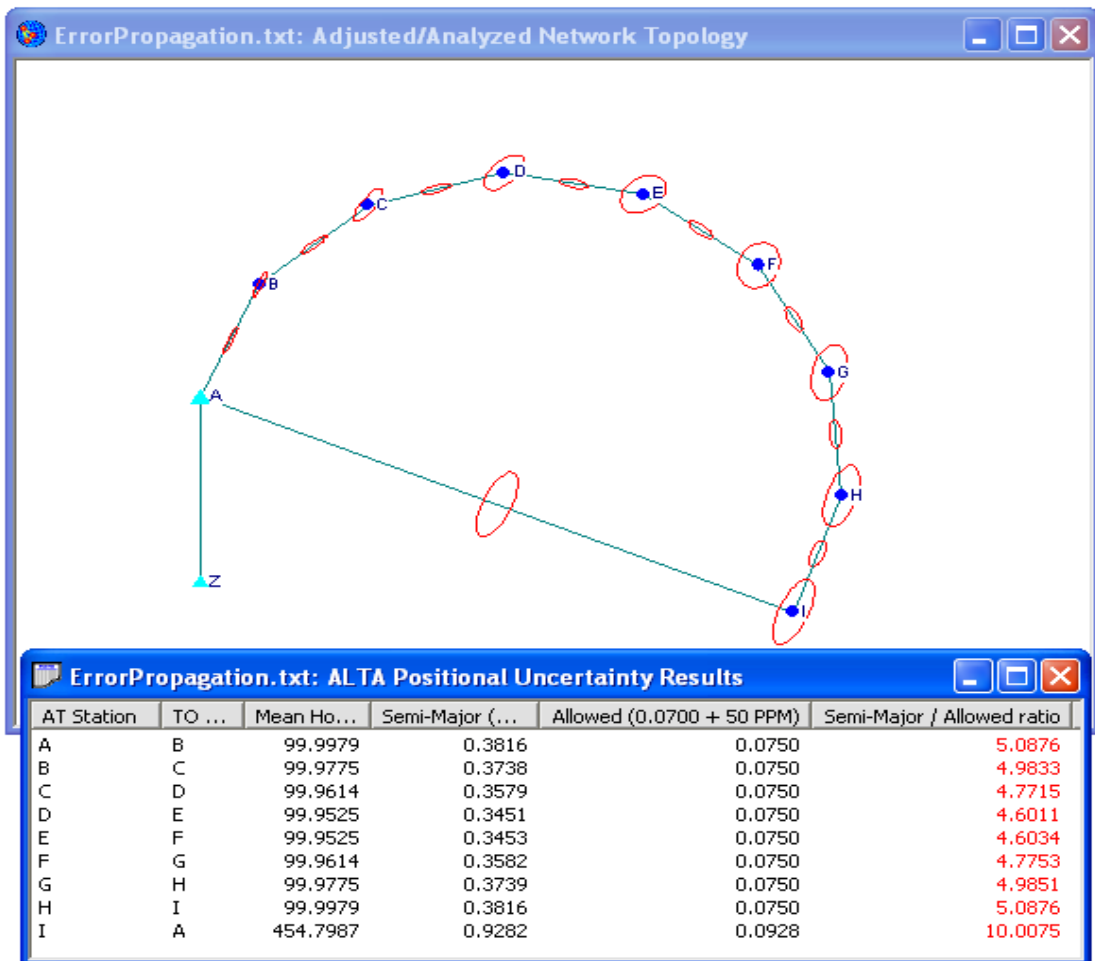
4. Computing The Error Ellipse

```
double atan2(double y, double x)
{
    if (x == 0.0)
    {
        if ( y == 0.0 )
            return 0.0; // x = y = 0.0
        else
            return sign(y) * Piidiv2;
    }
    else
    {
        if (x > 0.0) // First or Fourth quad
            return atan(y / x);
        else // Second or Third quad
            return sign(y)*(Pii-atan(fabs(y/x)));
    }
}
```

```
double convert_to_az(double ang)
{
    if (ang < 0.0)
        ang + Piitim2;
    else if (ang > Piitim2)
        return ang - Piitim2;
    else
        return ang;
}
```

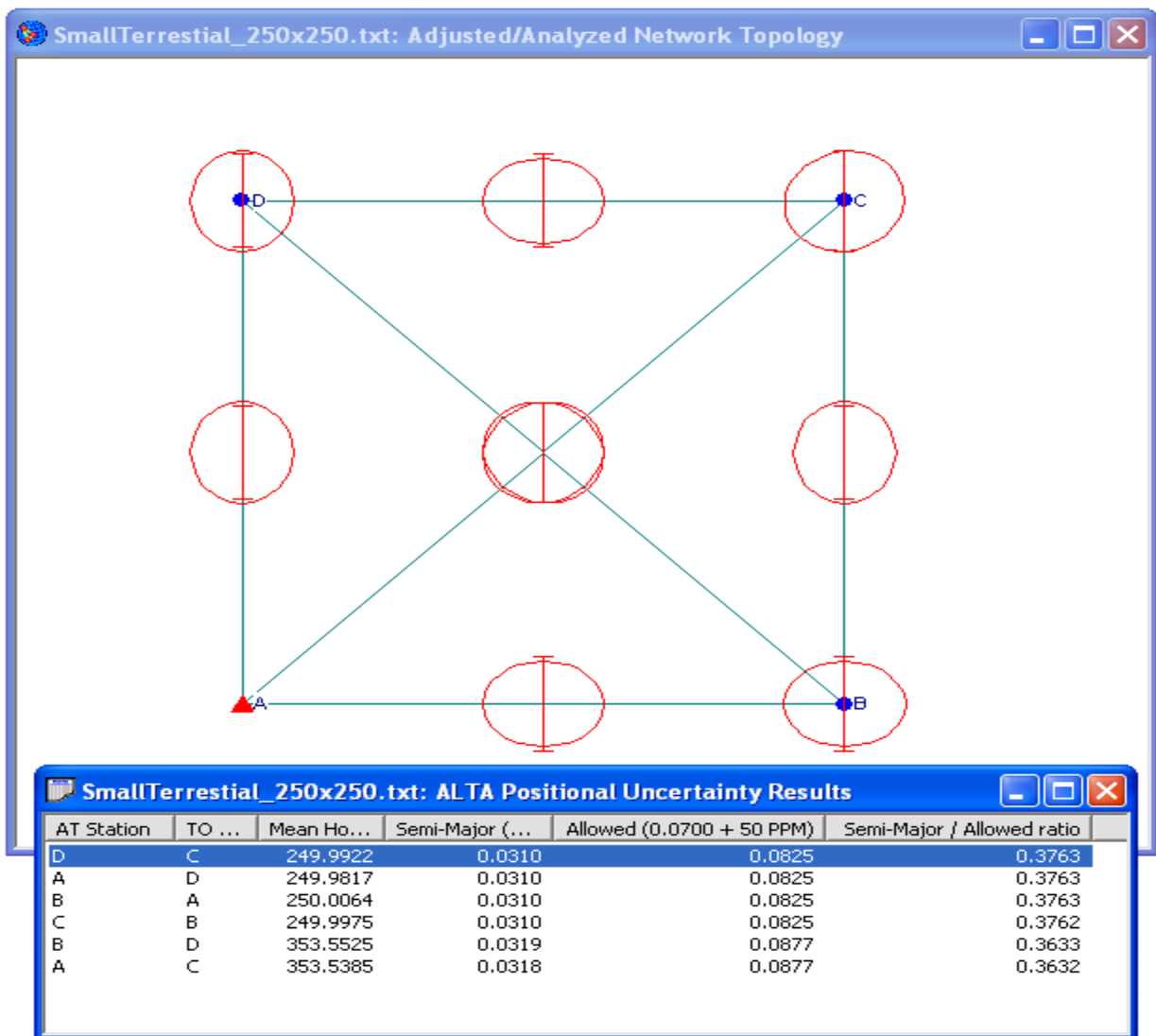
5. Sample Network Projects

One of the properties of random error is that it propagates as you move further from the known position. Although absolute error ellipses increase (left to right), the relative error ellipses do not increase much. The relative error ellipses reflect the error between the station pair; the common errors cancel each other out and are therefore eliminated.



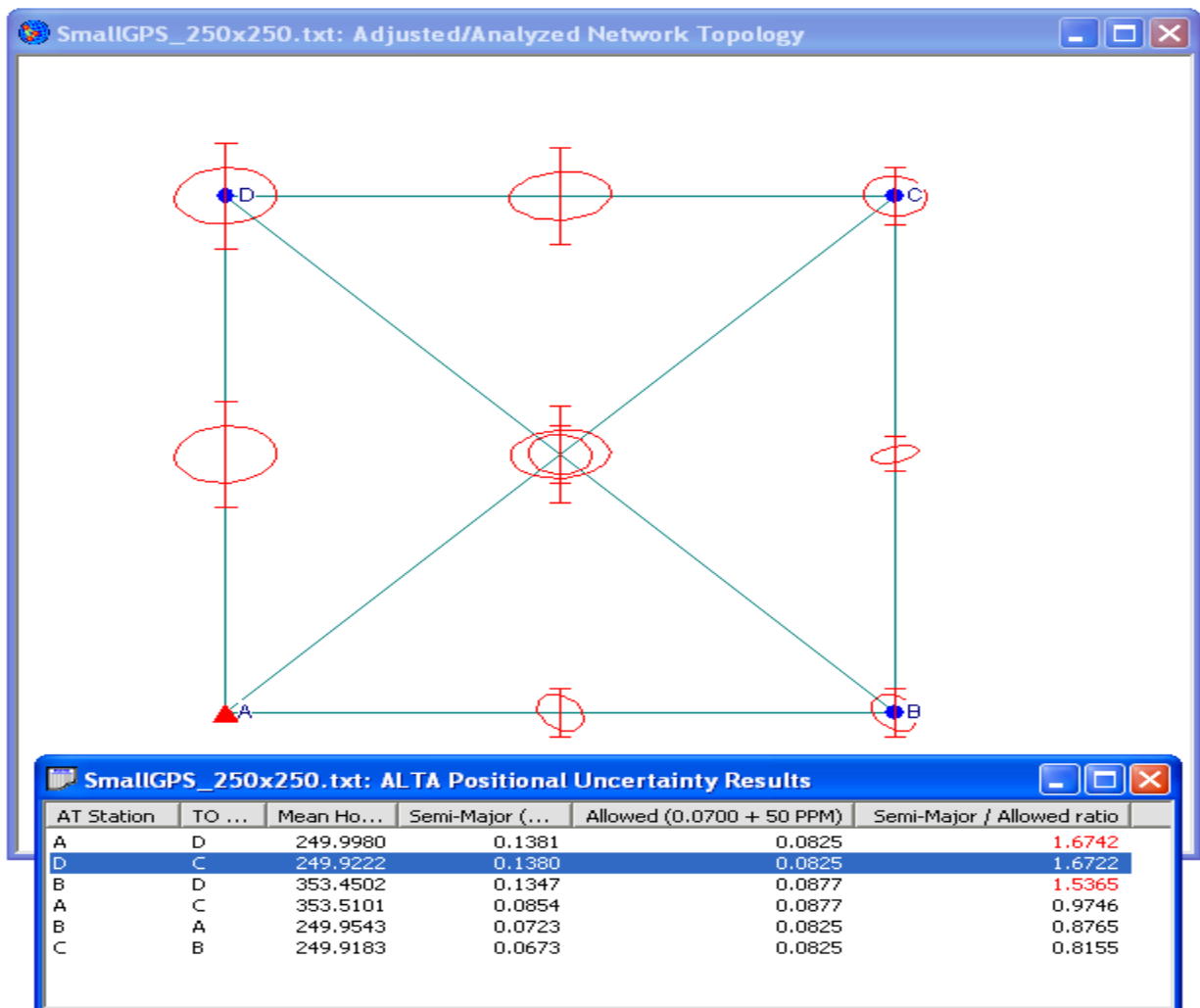
5. Sample Network Projects

For small projects, conventional measurements are often preferred due to their higher accuracy over short distances. This project covers an area of 250' by 250':



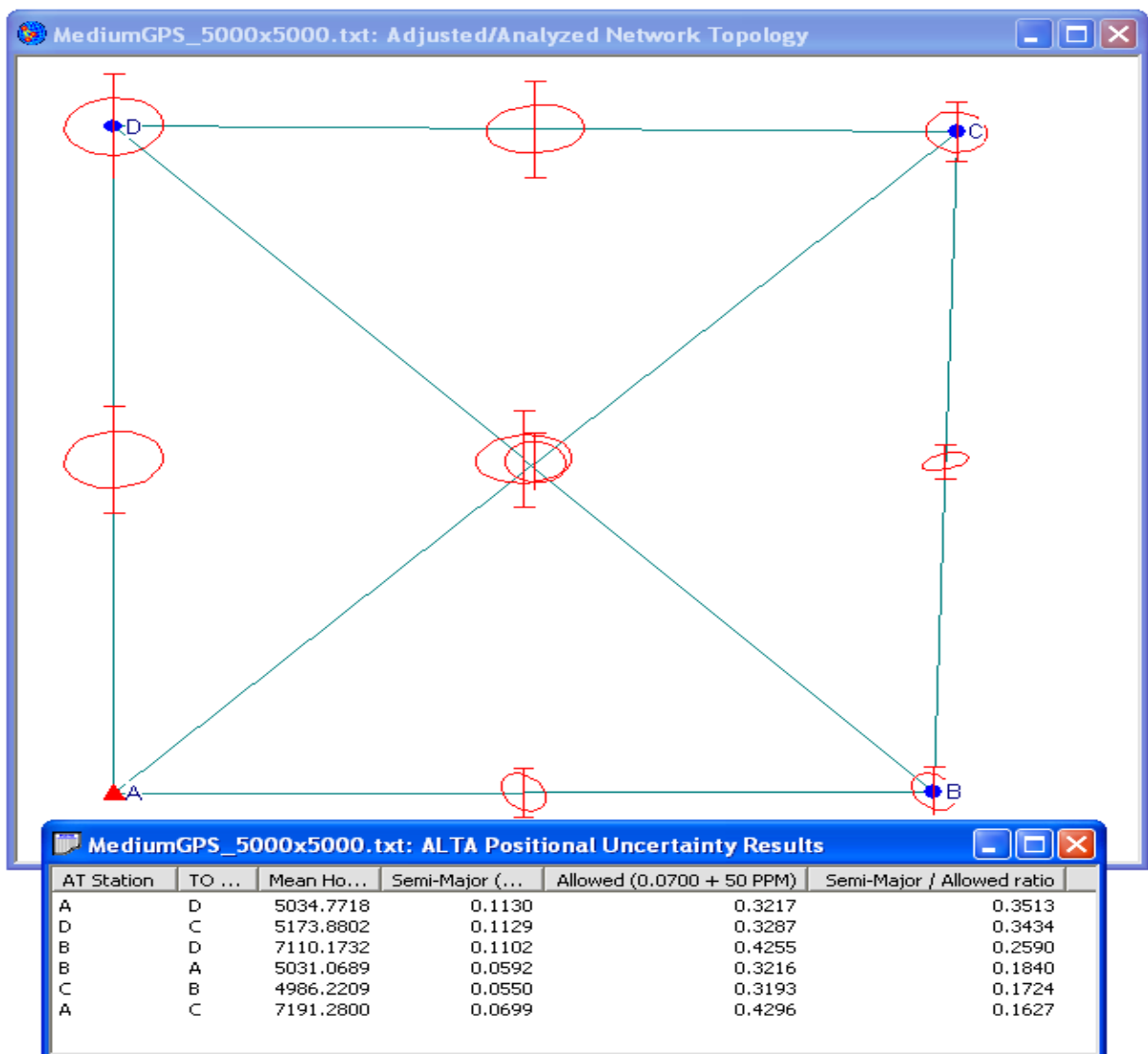
5. Sample Network Projects

Here is the same network using GPS. Notice that some of the ALTA tests fail. This is usually the result of larger minimum error of GPS. Since distances are short, allowable tolerance is small. This project covers an area of 250' by 250':



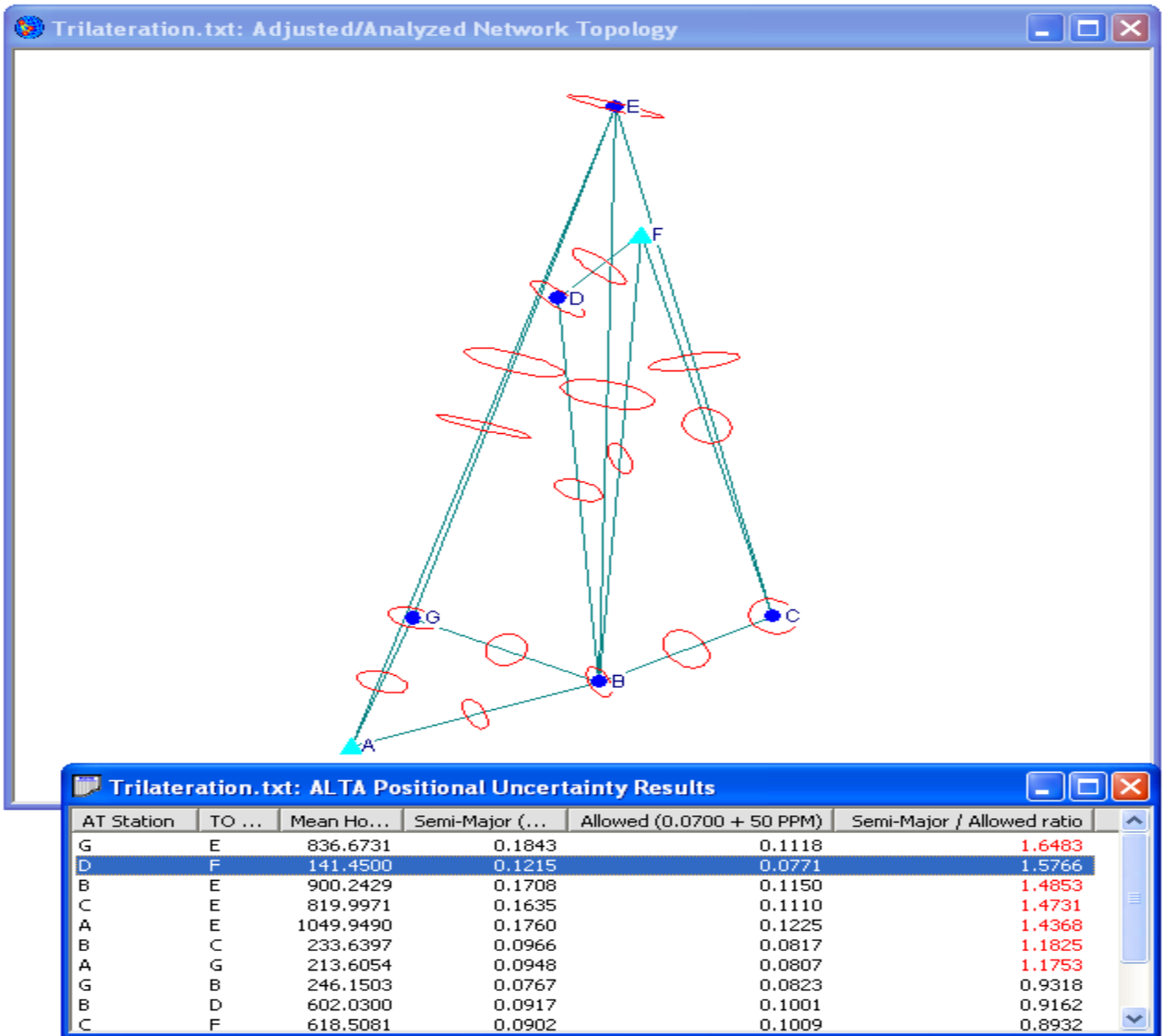
5. Sample Network Projects

This network has the same expected errors as the previous GPS network; however, the distances are longer and so are the allowable errors. These pass ALTA. This project covers an area of 5000' by 5000':



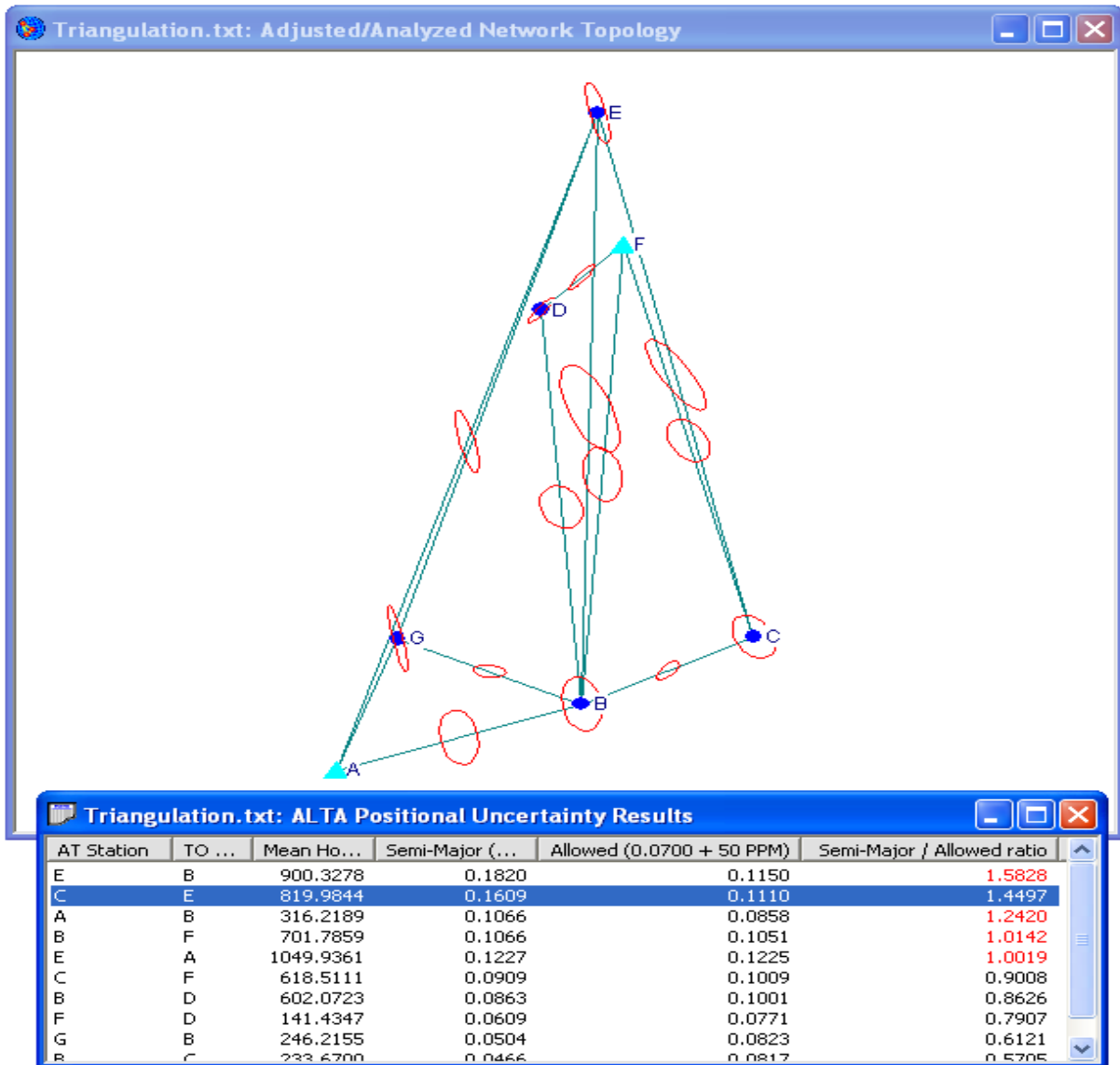
5. Sample Network Projects

This project contains only distances. Notice the direction of the uncertainty as shown by the shape and orientation of the error ellipses. Poor network geometry results in several failed tests.



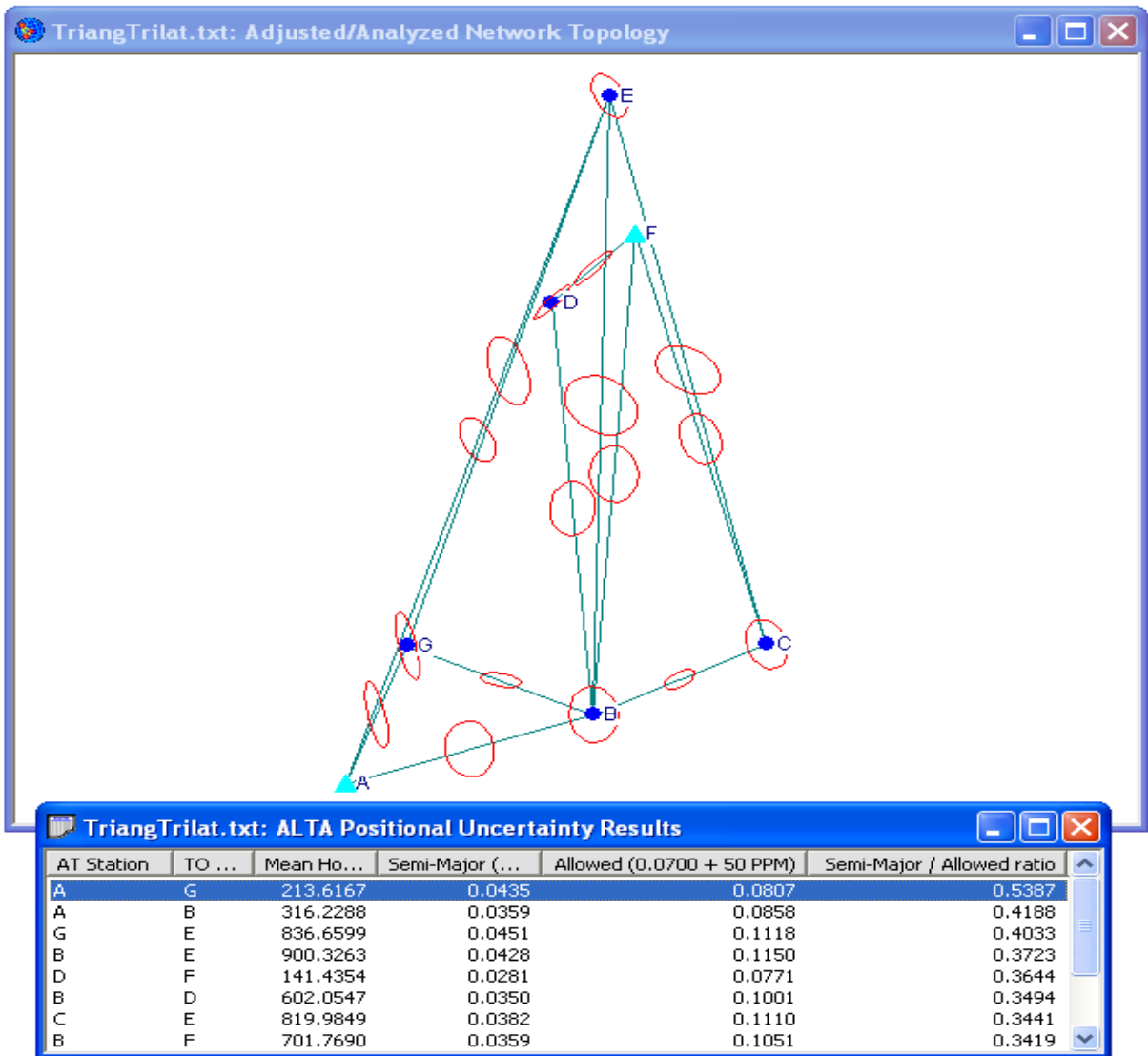
5. Sample Network Projects

Same network, but only angular measurements. Notice the direction of the error ellipses. Again, poor network geometry results in uncertainty and failed ALTA tests.



5. Sample Network Projects

Same shape as previous two projects, but now we have included all the distances and angles from these two projects. Even with same network geometry, all tests pass ALTA.



6. Network Design

Network design allows you to determine, in advance, if your survey will meet ALTA requirements. For network design, you need the following:

1. Approximate coordinates of surveyed locations.
2. The type of observations used between each station pair.
3. The expected error in your measurements (as expressed by the standard deviation of each observation).

If your design is adequate and you meet or exceed the expected error (in the field), the survey should pass ALTA testing.

SAMPLE Project

Stations: AAA, BBB, CCC, DDD.

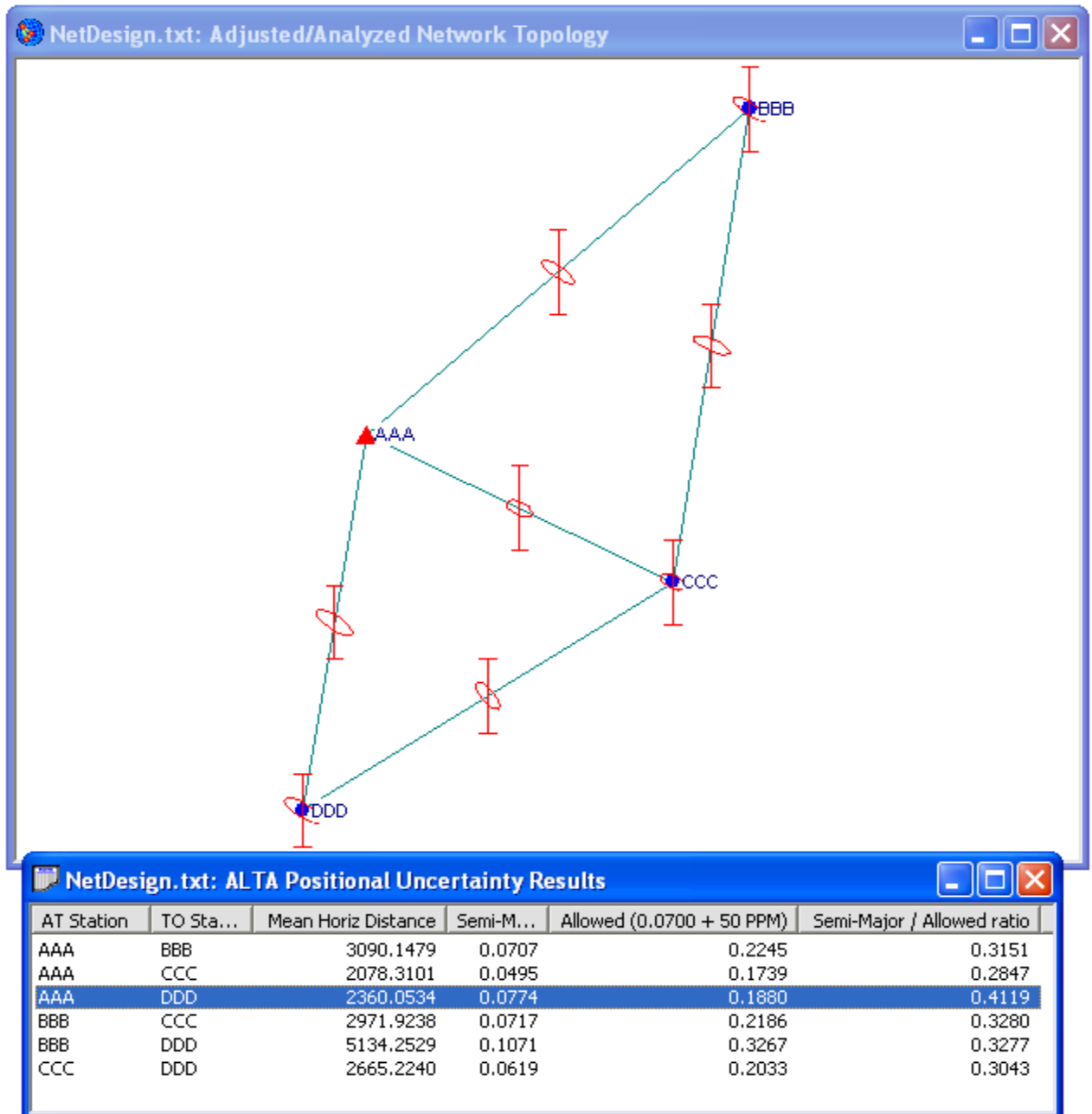
Hor Ang, Zen Ang, Slope Dist between each station.

SD's are 4.0, 7.0 sec and 0.01' respectively

Az from AAA to CCC for orientation (SD=1.0 sec).

Slope Dist from CCC to AAA (SD=0.01').

6. Network Design



7. Tips For Success

1. Equipment and crew must be well-functioning.
2. Utilize good network geometry: equilateral triangles for terrestrial observations and sky geometry for satellites.
3. Check field data daily, if possible. This makes it much easier to find blunders.
4. Use realistic standard deviations.
5. Eliminate systematic errors and blunders; use statistical tests to help identify outliers.
6. For large scale projects, use ellipsoidal height when adjusting in 3D. Using elevation can introduce a **one** PPM error for every six meters in height difference between the elevation value and the ellipsoidal height value.
7. Utilize Network Design as often as possible.